

# Stochastic Resonance

M.I. Dykman<sup>1</sup>, D.G. Luchinsky<sup>2</sup>, R. Mannella<sup>3</sup>,  
P.V.E. McClintock<sup>4</sup>, S.M. Soskin<sup>5</sup>, N.D. Stein<sup>4</sup> and N.G. Stocks<sup>6</sup>

<sup>1</sup>Department of Physics, Stanford University, Stanford, CA 94305, USA.

<sup>2</sup>VNIIMS, Andreevskaya nab 2, 117965 Moscow, Russia.

<sup>3</sup>Dipartimento di Fisica, Università di Pisa, Piazza Torricelli 2, 56100 Pisa, Italy.

<sup>4</sup>School of Physics and Materials, Lancaster University, Lancaster, LA1 4YB, UK.

<sup>5</sup>Institute of Semiconductor Physics, pr. Nauki 45, 252038 Kiev, Ukraine.

<sup>6</sup>Department of Engineering, Warwick University, Coventry, CV4 7AL, UK.

## Abstract

Stochastic resonance (SR) - a counter-intuitive phenomenon in which the signal due to a weak periodic force in a nonlinear system can be *enhanced* by the addition of external noise - is reviewed. A theoretical approach based on linear response theory (LRT) is described. It is pointed out that, although the LRT theory of SR is by definition restricted to the small signal limit, it possesses substantial advantages in terms of simplicity, generality and predictive power. The application of LRT to overdamped motion in a bistable potential, the most commonly studied form of SR, is outlined. Two new forms of SR, predicted on the basis of LRT and subsequently observed in analogue electronic experiments, are described.

## 1. Introduction

One of the most active current growth areas of nonlinear dynamics lies in the relatively unexplored region separating the two major divisions of the subject: that is, within the interface separating “deterministic” nonlinear dynamics, (e.g. Thompson and Stewart, 1986), where externally applied forces are precisely known (e.g. periodic), from stochastic nonlinear dynamics where the system under study fluctuates under the influence of a random force (e.g. Moss and McClintock, 1989). *Stochastic resonance* (SR), in which the signal due to a weak periodic force in a nonlinear system can, remarkably, be amplified by the addition of external noise, provides an example of a phenomenon in this interface region. It arises through a tripartite interaction between nonlinearity, fluctuations and a periodic force, and it cannot occur unless all three of these features are simultaneously present.

The notion of SR was originally introduced (Nicolis, 1982; Benzi et al, 1982) in relation to the earth’s ice-age cycle. The phenomenon was subsequently demonstrated in an electronic circuit (Fauve and Heslot, 1983) and in a ring laser (McNamara et al, 1988). Following this latter paper, there has been a veritable explosion of activity leading to the observation of SR or associated phenomena in a wide variety of contexts, including passive optical systems (Dykman et al, 1991), electron spin resonance (Gammaitoni et al, 1991a), sensory neurons (Longtin et al, 1991) and a magneto-elastic strip (Spano et al, 1992). These references are merely illustrative: a fuller bibliography can be found within the proceedings of a recent conference on SR (Moss, Bulsara and Shlesinger, 1993).

In this paper we introduce SR and set it within the context of classical linear response theory (LRT). We emphasize that the LRT perception of the phenomenon is very general. Not only does it provide a good description of SR in systems with static bistable potentials (conventional SR) but it also leads on naturally to the prediction of new forms of SR in quite different kinds of systems: see Wiesenfeld (1993). In Section 2 we describe this LRT approach and in Section 3 we show how it may be applied to conventional SR. Sections 4 and 5 describe two quite new forms of SR - associated with fluctuational transitions between coexisting periodic attractors, and for underdamped nonlinear oscillators in the absence of bistability - that were predicted on the basis of LRT and subsequently observed in electronic experiments. In Section 6 we summarise the results, discuss future directions, and draw conclusions.

## 2. Linear response theory of stochastic resonance

We shall define SR as an increase of the amplitude of a periodic signal in a nonlinear system resulting from the addition of external noise at the input; often, the signal/noise ratio at the output will also increase, an effect that meets the stricter definition of SR used by some authors. In both cases, the signal decreases again for sufficiently strong noise, giving rise to a resonance-like curve when the amplitude is plotted against noise intensity, thereby accounting for the terminology.

The theory of SR has been perceived as difficult, because of the need to treat stochastic and periodic forces together in a highly nonlinear system. It has mostly been developed with the simplifying assumption of a discrete two-state model (in the case of bistable systems) or, in the case of continuous systems, has been based on an approximate or numerical solution of the Fokker-Planck equation for a periodically driven system, sometimes with contradictory results (Benzi et al 1982; Nicolis, 1982; Presilla et al, 1989; Gammaitoni et al, 1989; McNamara and Wiesenfeld, 1989; Fox, 1989; Hu Gang, Nicolis and Nicolis, 1990; Jung and Hanggi, 1990, 1991).

The alternative approach to SR introduced by Dykman et al (1990a, 1990b), based on LRT, is quite different. According to LRT (see e.g. Landau and Lifshitz, 1980), if a system with coordinate  $q$  is driven by a weak force  $A \cos \Omega t$ , a small periodic term  $\delta \langle q(t) \rangle$  will appear in the ensemble-averaged value of the coordinate, oscillating at the same frequency  $\Omega$

$$\delta \langle q(t) \rangle = a \cos(\Omega t + \phi), \quad A \rightarrow 0 \quad (1)$$

$$a = A |\chi(\Omega)|, \quad \phi = -\arctan[\text{Im}\chi(\Omega)/\text{Re}\chi(\Omega)] \quad (2)$$

where  $\chi(\Omega)$  is the *susceptibility* of the system. The function  $\chi(\Omega)$  contains virtually everything needing to be known about the response of the system to a weak driving force. It gives both the *amplitude*  $a$  of the signal and its *phase lag*  $\phi$  relative to the driving force. The occurrence of a delta-shaped spike at frequency  $\Omega$  in the spectral density of fluctuations (SDF),  $Q(\omega)$ , of the system

$$Q(\omega) = \lim_{\tau \rightarrow \infty} (4\pi\tau)^{-1} \left| \int_{-\tau}^{\tau} dt q(t) \exp(i\omega t) \right|^2 \quad (3)$$

follows immediately from (1) on account of the principle of the decay of correlations

$$\langle q(t)q(t') \rangle \rightarrow \langle q(t) \rangle \langle q(t') \rangle \quad \text{for} \quad |t - t'| \rightarrow \infty$$

The *intensity*  $\frac{1}{4}a^2$  (i.e. area) of the spike can be found from (2). Following McNamara et al (1988), the signal/noise ratio in SR is often defined as the ratio  $R$  of the area of the spike to the value  $Q^{(0)}(\Omega)$  of the SDF at frequency  $\Omega$  but in the absence of the driving force. From (1) - (3), this quantity may be expressed in terms of the susceptibility as

$$R = \frac{1}{4}A^2|\chi(\Omega)|^2/Q^{(0)}(\Omega) \quad (A \rightarrow 0) \quad (4)$$

Consequently, the evolution of  $\chi(\Omega)$ , or of  $\chi(\Omega)$  and  $Q^{(0)}(\Omega)$ , with increasing noise intensity shows immediately whether or not SR in the signal or in the signal/noise ratio, respectively, is to be expected at frequency  $\Omega$  in any given system.

In the particular case of systems that are in thermal equilibrium, or quasi-equilibrium, the susceptibility at frequency  $\Omega$  can be obtained very simply from the fluctuation dissipation relations (Landau and Lifshitz, 1980),

$$\begin{aligned} \text{Re}\chi(\Omega) &= \frac{2}{T}P \int_0^\infty d\omega_1 Q^{(0)}(\omega_1)\omega_1^2(\omega_1^2 - \Omega^2)^{-1} \\ \text{Im}\chi(\Omega) &= \frac{\pi\Omega}{T}Q^{(0)}(\Omega) \end{aligned} \quad (5)$$

where  $P$  implies the Cauchy principal part, and  $T$  is the temperature (noise intensity) in energy units. It is interesting to note that a knowledge of  $Q^{(0)}(\omega)$  and its evolution with  $T$  is then sufficient in itself, to predict whether or not the system in question will exhibit SR: this would be true even where the underlying dynamics was unknown, and the information about  $Q^{(0)}(\omega)$  had been acquired by experiment.

### 3. Stochastic resonance in static double-well potentials

The initial tests of the above ideas were performed (Dykman et al, 1990a, 1990b) through the measurement of SDFs and the investigation of SR in an electronic model of the damped double-well Duffing oscillator,

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = A \cos \Omega t + f(t) \quad (6)$$

$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4, \quad \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 4\Gamma T\delta(t - t')$$

The results are shown in Fig 1, where the scaled signal/noise ratio  $\tilde{R} = 6.51 \times 10^{-4}R$  is plotted as a function of scaled noise intensity  $T/\Delta U$ ,  $\Delta U$  ( $= \frac{1}{4}$  for the potential in (6)) being the height of the potential barrier between the wells. The square data points represent direct measurements of  $\tilde{R}$ , obtained from the heights of the delta spikes in  $Q(\omega)$ ; the crosses are also obtained experimentally, but in a completely different way, from Equations (2), (4) and (5) using measurements of  $Q^{(0)}(\omega)$  in the absence of the periodic force. The fact that the agreement is excellent, within the experimental error, without the use of any adjustable parameters, can be regarded as a direct confirmation of the validity of the LRT perception of small signal SR.

For the particular parameters used for the measurements of Fig 1, the magnitude of the rise in  $\tilde{R}$  is relatively modest; much larger increases can be obtained for lower frequencies  $\Omega$  and larger damping constants  $\Gamma$ . Nonetheless, it is clear that there is a range of  $T/\Delta U$  within which  $\tilde{R}$  rises with increasing  $T$ , i.e. there is a manifestation of SR.

Of course, for a theory of SR, one would also need to be able to calculate  $Q^{(0)}(\omega)$ , rather than having to measure it experimentally. Although this has been done (Dykman et al, 1988) for the system (6), the most detailed experimental and theoretical studies relate to the equivalent overdamped system, which has been widely used as the standard system for investigations of SR,

$$\dot{q} + U'(q) = A \cos \Omega t + f(t) \quad (7)$$

$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4, \quad \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 2D\delta(t-t')$$

where  $f(t)$  is now a zero-mean Gaussian noise of intensity  $D$  (the way in which the overdamped limit is taken to obtain (7) from (6) is discussed e.g. by Risken (1989)). Like (6), (7) for  $A = 0$  is also a thermal equilibrium system so that, in order to find the susceptibility  $\chi(\omega)$ , it is only necessary to calculate the SDF  $Q^{(0)}(\omega)$  in the absence of the periodic force for substitution in the fluctuation dissipation relations (5) with  $T$  replaced by  $D$ . In the limit of weak noise,  $D \ll \Delta U$ , both  $Q^{(0)}(\omega)$  and  $\chi(\omega)$  can be obtained analytically (Dykman and Krivoglaз, 1979, 1984; Dykman et al 1989) as a sum of partial contributions from fluctuations about the equilibrium positions  $q_n$  and from interwell transitions,

$$Q^{(0)}(\omega) = \sum_{n=1,2} w_n Q_n^{(0)}(\omega) + Q_{tr}^{(0)}(\omega), \quad \chi(\omega) = \sum_{n=1,2} w_n \chi_n(\omega) + \chi_{tr}(\omega) \quad (8)$$

Here  $w_n$  is the population of the  $n$ th stable state and, for the model (7),  $w_1 = w_2 = \frac{1}{2}$ ,  $Q_1^{(0)}(\omega) = Q_2^{(0)}(\omega)$  and  $\chi_1(\omega) = \chi_2(\omega)$ . The SDF for the intrawell vibrations  $Q_n^{(0)}(\omega)$  is obtained by expanding  $U(q)$  about the equilibrium position;  $Q_{tr}^{(0)}(\omega)$  can be written down in terms of the transition probabilities  $W_{nm}^{(0)}$ , defining the probability of an  $n \rightarrow m$  transition in the absence of periodic forcing.

These calculations result in explicit analytic predictions for  $R(D)$  and  $\phi(D)$ . The latter is of particular interest in view of the inconsistent results of earlier calculations and experiments, with (Nicolis, 1982; McNamara and Wiesenfeld, 1989; Hu Gang et al 1990) on the one hand, and (Gammaitoni et al, 1990, 1991 a, b) on the other. An analogue electronic experiment was performed (Dykman et al, 1992a) to measure  $\phi(D)$  for comparison with the LRT theoretical predictions, yielding the results shown by the data points in the main section of Fig 2; the inset shows a plot of  $R/A^2$  as a function of  $D$  in the range near the minimum where other theories fail. In both cases, the agreement between experiment and theory is very satisfactory, providing further confirmation of the validity of the LRT approach to SR. The dashed line shows the prediction of earlier (two-state) theories (e.g. Nicolis, 1982; McNamara and Wiesenfeld, 1989) that do not include the effect of intrawell motion.

#### 4. Stochastic resonance for periodic attractors

It is clear from the above discussion that any system whose susceptibility  $\chi(\Omega)$  increases with noise intensity may be expected to display SR when driven by a weak periodic force of frequency  $\Omega$ . Dykman and Krivoglaz (1979) had noticed such an effect in the imaginary part of the susceptibility of a periodically driven nonlinear oscillator with coexisting periodic attractors. It was therefore obvious that SR was to be expected in systems of this kind, and that it would be likely to have some unusual and characteristic features distinguishing it from conventional SR. The new phenomenon has been sought and recently found and investigated (Dykman et al, 1993 b, c). The results have implications for a large class of passive optically bistable systems and, in particular, for optically bistable microcavities.

The system that we consider is the nearly-resonantly-driven, underdamped, single-well Duffing oscillator with additive noise,

$$\ddot{q} + 2\Gamma\dot{q} + \omega_0^2 q + \gamma q^3 = F \cos(\omega_F t) + f(t) \quad (9)$$

$$\Gamma, |\delta\omega| \ll \omega_F, \quad \gamma\delta\omega > 0, \quad \delta\omega = \omega_F - \omega_0, \quad \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 4\Gamma T\delta(t - t')$$

Note that the force  $F \cos(\omega_F t)$  is not very weak; neither is it so strong that the system becomes chaotic or displays subharmonics. Within a certain parameter range, (9) is characterised by two coexisting periodic attractors of different amplitude and phase. Weak noise  $f(t)$  causes occasional transitions between them. For appropriate noise intensity, these transitions can become coherent on average with an additional weak periodic trial force  $A \cos(\Omega t + \phi)$  added to (9), provided that  $\Omega$  is close to  $\omega_F$ , leading to stochastic amplification, i.e. SR. It can be shown that the system responds strongly to the trial force, not only at  $\Omega$  but also at  $|2\omega_F - \Omega|$ ; the relevant susceptibilities can be calculated by an extension of the Dykman and Krivoglaz (1979) theory.

Measurements of the signal/noise ratio  $R$  in an analogue electronic experiment (data points) are compared with the theoretical predictions in Fig 3. Although the results are very similar to those found in conventional SR (Moss et al, 1993), it must be emphasized that SR for periodic attractors also possesses a number of features that are entirely different. The most important of these are, first, that it is a high frequency phenomenon. Stochastic amplification takes place not at a low frequency comparable to the inter-attractor hopping rate, as in conventional SR, but at the much higher frequency  $\Omega$  comparable to  $\omega_0$ . Secondly the stochastic enhancement of the signal at the mirror-reflected frequency  $|\Omega - 2\omega_F|$  has no analogue in conventional SR. Other differences, and the relationship to phenomena in nonlinear optics, are discussed by Dykman et al (1993 c).

#### 5. Stochastic resonance in monostable systems

Until recently, it was the almost universal assumption (Moss et al, 1993, and references therein) that the stochastic amplification of SR could occur only as the result of nearly periodic fluctuational transitions between coexisting attractors, corresponding to the minima of a static bistable potential. The high-frequency SR of Section 4 extends the picture to encompass periodic attractors, but it still requires bistability. The LRT picture of SR, however, does not involve any such requirement: any system, bistable or

otherwise, in which the susceptibility  $\chi(\Omega)$  increases with noise intensity would, in view of (1), (2), be expected to display SR.

One monostable system in which SR is to be anticipated on these grounds is the underdamped, single-well, Duffing oscillator subject to a constant field

$$\ddot{q} + 2\Gamma\dot{q} + U'(q) = A \cos \Omega t + f(t) \quad (10)$$

$$U(q) = \frac{1}{2}q^2 + \frac{1}{4}q^4 + Bq, \quad \Gamma \ll 1, \quad \langle f(t) \rangle = 0, \quad \langle f(t)f(t') \rangle = 4\Gamma T \delta(t - t')$$

which is known (Stocks et al, 1993a) to have extremely sharp zero-dispersion peaks (ZDPs) in its SDFs provided that  $|B| > 8/(7)^{3/2}$  so that the variation of the oscillator's eigenfrequency  $\omega(E)$  with energy  $E$  possesses an extremum (Dykman et al, 1990c). The ZDPs rise exponentially fast with increasing noise intensity  $T$ . Thus, because (13) for  $A = 0$  is a system of the thermal equilibrium type, to which (5) is applicable,  $\chi(\Omega)$  may also be expected to rise extremely fast provided that  $\Omega$  is chosen to be in the close vicinity of the ZDP.

Experimental results (Stocks et al, 1993b) obtained from an analogue electronic model of (10) with  $B = 2$  are shown by the circle data of Fig 4(a). The quantity plotted is the square of the stochastic amplification factor

$$S(T) = a(T)/a(0)$$

where  $a(T)$  is the amplitude of the signal for noise intensity  $T$ . The fact that  $S(T)$  rises very rapidly (from the value of unity that it would take in the absence of stochastic amplification) provides a clear signature of SR. The fuller curve is the LRT prediction, based on (1) and (5) using the expressions for  $Q^{(0)}(\omega)$  given by Dykman et al (1990c). It is in very satisfactory agreement with the data. The phase shift  $\phi(T)$  has also been measured and calculated, as shown by the circle data and associated curve in Fig 4(b). Here, too, experiment and LRT theory agree well. A physically motivated discussion (together with an explanation of the experimental and theoretical results obtained for  $B = 0$  shown by the square data and associated curves) of this new form of SR has been given by Stocks et al (1993b). It can be demonstrated (Stocks et al, 1992) on the basis of LRT that, for sufficiently small  $\Gamma$  in (10), substantial increases, not only in the signal, but also in the signal/noise ratio  $R$  are to be expected.

## 6. Conclusion

We conclude that LRT provides a good description, not only of conventional SR, but also of the other new forms of SR that can be predicted on that basis. These include SR for periodically modulated noise (Dykman et al 1992b) which has not been considered here, as well as the SR for periodic attractors and monostable systems discussed above. Although LRT is, by definition, applicable only in the small signal limit, in combination with the corresponding physical picture of SR it provides a valuable clue as to the general type of behaviour to be expected even for larger amplitudes of the trial force.

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## Figure Captions

1. Signal/noise ratio  $\tilde{R}$  for the electronic model of the double-well oscillator (6), as a function of scaled noise intensity  $T/\Delta U$ . Direct measurements of  $\tilde{R}$  (squares) are compared with results calculated from measured spectra  $Q^{(0)}(\omega)$  (crosses) using the fluctuation dissipation relations (5). (After Dykman et al, 1990a).
2. The phase shift  $-\phi$  (degrees) between the periodic force and the response measured (data points) for an electronic model of the overdamped double-well system (7) with  $\Omega = 0.1$  and  $A = 0.04$  (circles) and  $A = 0.2$  (squares). The full curve represents the LRT theory based on (5); the dashed curve represents the prediction of earlier 2-state theories. Inset: the normalised signal/noise ratio as a function of noise intensity  $D$ , showing that the LRT theory works well near the minimum. (After Dykman et al, 1990a).
3. The signal/noise ratio  $R$  of the response of the system (9) to a weak trial force at frequency  $\Omega$ , as a function of noise intensity  $\alpha$ , in experiment and LRT theory: at the trial frequency  $\Omega$  (circle data and associated curve); and at the mirror-reflected frequency  $|2\omega_F - \Omega|$  (squares). For noise intensities near and beyond the maxima in  $R(\alpha)$ , the asymptotic theory is only qualitative and so the curves are shown dotted. After Dykman et al, 1990c).
4. (a) The squared stochastic amplification factor  $S^2$  measured for the electronic model of the system (10) with  $|B| = 2$ ,  $A = 0.02$ , (circle data) compared with the LRT theory (curve) based on the fluctuation dissipation relations (5). (After Stocks et al (1993b) where the square data points and associated curve, for  $B = 0$ , are also discussed.)